

Polynomial Hopf algebras in Algebra & Topology

Andrew Baker
University of Glasgow/MSRI

UC Santa Cruz Colloquium

6th May 2014

last updated 07/05/2014

Graded modules

Given a commutative ring \mathbb{k} , a *graded \mathbb{k} -module* $M = M_*$ or $M = M^*$ means sequence of \mathbb{k} -modules M_n or M^n . In practise we will always have $M_n = 0$ or $M^n = 0$ whenever $n < 0$ so M is *connective*. We will usually drop the word graded!

If $x \in M_n$ or $x \in M^n$ then n is the *degree* of x and we set $|x| = n$. It is useful to view an ungraded \mathbb{k} -module N as graded with $N_0 = N = N^0$ and $N_n = 0 = N^n$ whenever $n \neq 0$.

We can form tensor products of such graded modules by setting

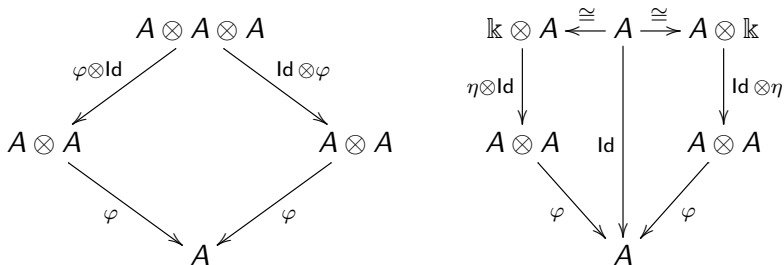
$$(M \otimes_{\mathbb{k}} N)_n = \bigoplus_i M_i \otimes_{\mathbb{k}} N_{n-i}$$

and so on. We usually write \otimes for $\otimes_{\mathbb{k}}$.

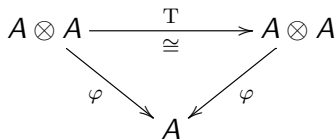
There is a switch isomorphism $T: M \otimes N \xrightarrow{\cong} N \otimes M$ for which

$$T(x \otimes y) = (-1)^{|x||y|} y \otimes x.$$

A (*connected*) \mathbb{k} -algebra A_* or A^* is a connective \mathbb{k} -module with $A_0 = \mathbb{k}$ or $A^0 = \mathbb{k}$, and a \mathbb{k} -linear *product* $\varphi: A \otimes A \rightarrow A$, i.e., a sequence \mathbb{k} -homomorphisms $A_i \otimes A_j \rightarrow A_{i+j}$, fitting into some commutative diagrams.

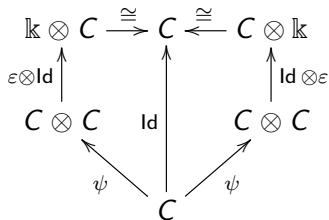
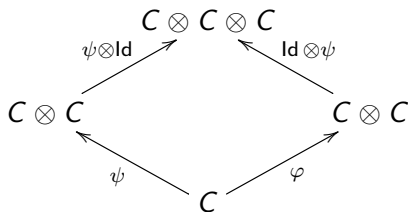


Here the unit homomorphism $\eta: \mathbb{k} \rightarrow A$ is the inclusion of \mathbb{k} as A_0 or A^0 . A is *commutative* if the following diagram commutes.

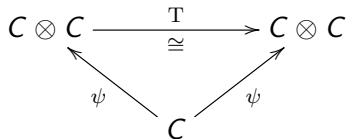


Coalgebras

The *dual* notion is that of a (*connected*) *coalgebra*, which is a triple (C, ψ, ε) , with C a connected \mathbb{k} -module, $\psi: C \rightarrow C \otimes C$, and $\varepsilon: C \rightarrow \mathbb{k}$ trivial except in degree $n = 0$ in which case it is an isomorphism, and this data fits into some commutative diagrams.



If the following diagram commutes then C is *cocommutative*.



Examples

Let x have degree $d \in \mathbb{N}$. Then the *free \mathbb{k} -algebra* $\mathbb{k}\langle x \rangle$ has

$$\mathbb{k}\langle x \rangle_{kd} = \mathbb{k}\langle x \rangle^{kd} = \mathbb{k}\{x^k\},$$

and is trivial in degrees not divisible by d . The *free commutative \mathbb{k} -algebra* $\mathbb{k}[x]$ is the quotient algebra $\mathbb{k}\langle x \rangle / (x^2 - (-1)^d x^d)$. When $\text{char } \mathbb{k} = 2$, $\mathbb{k}[x] = \mathbb{k}\langle x \rangle$, but if $2 \in \mathbb{k}^\times$ and d is odd, $\mathbb{k}[x] = \mathbb{k}\langle x \rangle / (x^2)$ is an *exterior algebra*. This generalises to free commutative algebras on collections of elements x_α of positive degrees. If all generators are in even degrees then we get a *polynomial algebra*

$$\mathbb{k}[x_\alpha : \alpha] = \bigotimes_{\alpha} \mathbb{k}[x_\alpha],$$

if they are all in odd degrees then we get an *exterior algebra*

$$\mathbb{k}[x_\alpha : \alpha] = \Lambda_{\mathbb{k}}(x_\alpha : \alpha).$$

The *free algebra* on a collection of elements y_β is built out of the tensor powers of the free module $Y = \mathbb{k}\{y_\beta : \beta\}$.

For some basic coalgebras, we can take an indeterminate y of even degree $2d$ and $C = \mathbb{k}[y]$. For $\psi: C \rightarrow C \otimes C$ take the *Binomial coproduct*

$$\psi(y^k) = \sum_{i=0}^k \binom{k}{i} y^i \otimes y^{k-i},$$

and also set $\varepsilon(y^k) = 0^k$.

For a more interesting version, take $C_{2k} = \mathbb{k}\{y^{[k]}\}$ and the *Leibnitz coproduct*

$$\psi(y^{[k]}) = \sum_{i=0}^k y^{[i]} \otimes y^{[k-i]}.$$

If $\text{char } \mathbb{k} = 0$ then we can think of $y^{[k]}$ as $y^k/k!$, but the above makes sense for any \mathbb{k} .

Hopf algebras

Suppose that (A, φ, η) is an algebra and (A, ψ, ε) is a coalgebra. Then $(A, \varphi, \eta, \psi, \varepsilon)$ is a Hopf algebra if either of the following holds:

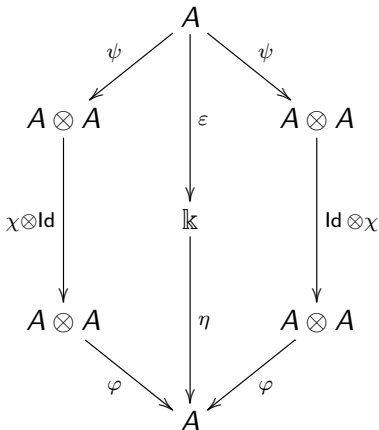
- ▶ (A, φ, η) is commutative and ψ, ε are algebra homomorphisms;
- ▶ (A, ψ, ε) is cocommutative and φ, η are coalgebra homomorphisms.

Note that in the first case φ, η are algebra homomorphisms, while in the second, ψ, ε are coalgebra homomorphisms. Here the tensor product of algebras A_1, A_2 is given the product

$$(A_1 \otimes A_2) \otimes (A_1 \otimes A_2) \xrightarrow{\cong} (A_1 \otimes A_1) \otimes (A_2 \otimes A_2) \xrightarrow{\varphi_1 \otimes \varphi_2} A_1 \otimes A_2$$

and similarly for coalgebras. So these notions are even more symmetric than might appear. The Hopf algebra is called *bicommutative* if both algebra and coalgebra structures are commutative.

If A is a commutative algebra or a commutative coalgebra, the connectivity assumption forces the existence of an *antipode* $\chi: A \rightarrow A$ which is an involution that is both an algebra and a coalgebra anti-isomorphism making the following diagram commutative.



The symmetric function Hopf algebra

Take generators $c_n \in \text{Symm}^{2n}$ and form the polynomial algebra $\text{Symm} = \mathbb{k}[c_n : n \geq 1]$. Notice that the free module $\mathbb{k}\{c_n : n \geq 0\}$ is also a cocommutative coalgebra with the Leibnitz coproduct.

Theorem

Symm is the free bicommutative Hopf algebra generated by the cocommutative coalgebra $\mathbb{k}\{c_n : n \geq 0\}$.

Define the *dual* of Symm by taking the \mathbb{k} -linear dual

$$\text{Symm}_n = \text{Hom}_{\mathbb{k}}(\text{Symm}^n, \mathbb{k})$$

and taking the adjoints φ_*, ψ_* to be the compositions

$$\begin{aligned} \text{Symm}_* \otimes \text{Symm}_* &\xrightarrow{\cong} \text{Hom}_{\mathbb{k}}(\text{Symm} \otimes \text{Symm}, \mathbb{k}) \\ &\xrightarrow{\psi^\dagger} \text{Hom}_{\mathbb{k}}(\text{Symm}, \mathbb{k}) = \text{Symm}_*, \end{aligned}$$

$$\begin{aligned} \text{Symm}_* &= \text{Hom}_{\mathbb{k}}(\text{Symm}, \mathbb{k}) \xrightarrow{\varphi^\dagger} \text{Hom}_{\mathbb{k}}(\text{Symm} \otimes \text{Symm}, \mathbb{k}) \\ &\xrightarrow{\cong} \text{Symm}_* \otimes \text{Symm}_*. \end{aligned}$$

We also define $\eta^* = \varepsilon^\dagger$ and $\varepsilon^* = \eta^\dagger$. Then $(\text{Symm}_*, \varphi^*, \eta^*, \psi^*, \varepsilon^*)$ is a bicommutative Hopf algebra.

Theorem

There is an isomorphism of Hopf algebras $\text{Symm}^ \cong \text{Symm}_*$, hence Symm^* is self dual.*

Corollary

Symm_ is a polynomial algebra.*

Under this isomorphism $\text{Symm}^* \cong \text{Symm}_*$, let $c_n \leftrightarrow b_n$. We can also try to understand elements of Symm_* in terms of duality. If we use the monomial basis $c_1^{r_1} \cdots c_\ell^{r_\ell}$ then the dual of the monomial c_1^k is b_n , while the dual of the indecomposable c_n is an element q_n which satisfies $\psi_*(q_n) = q_n \otimes 1 + 1 \otimes q_n$ so it is *primitive*. In fact the primitive module in degree $2n$ is generated by q_n ,

$$\text{Pr Symm}_{2n} = \mathbb{k}\{q_n\}$$

and the Newton recurrence formula is satisfied:

$$q_n = b_1 q_{n-1} - b_2 q_{n-1} + \cdots + (-1)^{n-2} b_{n-1} q_1 + (-1)^{n-1} n b_n.$$

Under the isomorphism there is also a primitive s_n in Symm^{2n} . There is a self dual basis consisting of Schur functions $S_\mu(c_1, \dots)$ which are dual to the $S_\mu(b_1, \dots)$. The s_n and q_n are special cases of these.

The structure of Sym is sensitive to the ring \mathbb{k} . For example, if $\mathbb{k} = \mathbb{Q}$, there is a decomposition of Hopf algebras

$$\text{Sym}^* = \bigotimes_{n \geq 1} \mathbb{Q}[s_n].$$

Let p be a prime and let $\mathbb{k} = \mathbb{F}_p$ or $\mathbb{k} = \mathbb{Z}_{(p)}$. There is a decomposition of Hopf algebras

$$\text{Sym}^* = \bigotimes_{p \nmid m} B[2m],$$

where

$$B[2m] = \mathbb{k}[a_{m,r} : r \geq 0]$$

is an indecomposable polynomial Hopf algebra and

$$s_{mp^r} = p^r a_{m,r} + p^{r-1} a_{m,r-1}^p + \cdots + p a_{m,1}^{p^{r-1}} + a_{m,0}^{p^r}.$$

This connection with Witt vectors leads to Sym being viewed as the *big Witt vector* Hopf algebra.

Occurrences of Symm in nature

One interpretation of Symm^{2n} is as the \mathbb{k} -module of homogeneous symmetric functions of degree n in k indeterminates t_i where $k \geq n$. It is a classical result that this is correct and then c_n corresponds to the elementary function $\sum t_1 \cdots t_n$, while s_n corresponds to the power sum $\sum t_1^n$.

We can also identify Symm^{2n} with the representation/character ring of the symmetric group Σ_n , $R(\Sigma_n)$ under addition. Then $R = \bigoplus_{n \geq 0} R(\Sigma_n)$ has a Hopf algebra structure agreeing with that of Symm and it is also self dual under inner product of characters.

In Algebraic Topology, Symm^* appears as $H^*(BU; \mathbb{k})$, the cohomology ring of the classifying space BU . The coproduct is induced from the map $BU \times BU \rightarrow BU$ classifying direct summand of vector bundles. Here c_n is the universal n -th Chern class and the coproduct is equivalent to the Cartan formula.

Dually, $\text{Symm}_* = H_*(BU; \mathbb{k})$.

A non-commutative analogue

Starting with the Leibnitz cocommutative coalgebra $\mathbb{k}\{z_n : n \geq 0\}$ where $|z_n| = 2n$. We can form the free algebra generated by the z_n with $n \geq 1$, $\text{NSymm}_* = \mathbb{k}\langle z_n : n \geq 1 \rangle$. It has a basis consisting of ordered monomials $z_{r_1} \cdots z_{r_\ell}$. The Leibnitz coproduct extends, e.g.,

$$\psi_*(z_m z_n) = \sum_{i,j} z_i z_j \otimes z_{m-i} z_{n-j}.$$

The counit is given by $\varepsilon_*(z_k) = 0$ if $k > 0$ and $\varepsilon_*(z_0) = \varepsilon_*(1) = 1$.

Theorem

NSymm_* is a cocommutative Hopf algebra.

The ring of quasi-symmetric functions QSymm^* is the dual, $\text{QSymm}^n = \text{Hom}_{\mathbb{k}}(\text{NSymm}_n, \mathbb{k})$.

Theorem

QSymm^* is a commutative Hopf algebra.

Ditters Conjecture ca 1972: QSymm^* is a polynomial ring. (First apparently correct proof by Hazewinkel 2000).

The product in QSymm is complicated. If we denote by $[r_1, \dots, r_\ell]$ the dual to the monomial $z_{r_1} \cdots z_{r_\ell}$ then products involve *overlapping shuffles*. For example,

$$\begin{aligned} [1, 2][3] &= [1, 2, 3] + [1, 3, 2] + [3, 1, 2] + [1, 2 + 3] + [1 + 3, 2] \\ &= [1, 2, 3] + [1, 3, 2] + [3, 1, 2] + [1, 5] + [4, 2]. \end{aligned}$$

In NSymm_{*} there are many primitives in each degree. For example,

$$\begin{aligned} q'_n &= z_1 q'_{n-1} - z_2 q'_{n-1} + \cdots + (-1)^{n-2} z_{n-1} q'_1 + (-1)^{n-1} n z_n, \\ q''_n &= q''_{n-1} z_1 - q''_{n-1} z_2 + \cdots + (-1)^{n-2} q''_1 z_{n-1} + (-1)^{n-1} n z_n, \end{aligned}$$

define two different families of primitives. This makes it difficult to understand the indecomposables in QSymm^{*}.

Topology to the rescue!

In fact, these Hopf algebras appear in topological disguise:

$$\mathrm{NSymm}_* \cong H_*(\Omega\Sigma\mathbb{C}P^\infty), \quad \mathrm{QSymm}^* \cong H^*(\Omega\Sigma\mathbb{C}P^\infty).$$

Theorem (Topological proof: Baker & Richter 2006)

$H^*(\Omega\Sigma\mathbb{C}P^\infty; \mathbb{Z})$ is polynomial.

I will outline an approach to proving this which differs from the original one but extends to many other examples in a uniform way. To simplify things I'll only concentrate on the case of a field \mathbb{k} , the most interesting example being $\mathbb{k} = \mathbb{F}_p$ for a prime p . The rational case is easy and we have a local to global argument for the case $\mathbb{k} = \mathbb{Z}$.

The Eilenberg-Moore spectral sequence

Let \mathbb{k} be a field and write $H^*(-) = H^*(-; \mathbb{k})$. We also need to assume that X is simply connected.

Theorem

There is a second quadrant Eilenberg-Moore spectral sequence of \mathbb{k} -Hopf algebras $(E_r^{,*}, d_r)$ with differentials*

$$d_r: E_r^{s,t} \longrightarrow E_r^{s+r, t-r+1}$$

and

$$E_2^{s,t} = \mathrm{Tor}_{H^*(X)}^{s,t}(\mathbb{k}, \mathbb{k}) \implies H^{s+t}(\Omega X).$$

The grading conventions here give

$$\mathrm{Tor}_{H^*(X)}^{s,*} = \mathrm{Tor}_{-s,*}^{H^*(X)}$$

in the standard homological grading.

When $\mathbb{k} = \mathbb{F}_p$ for a prime p , this spectral sequence admits Steenrod operations. We will denote the mod p Steenrod algebra by $\mathcal{A}^* = \mathcal{A}(p)^*$.

Theorem

If $H^(-) = H^*(-; \mathbb{F}_p)$ for a prime p , then the Eilenberg-Moore spectral sequence is a spectral sequence of \mathcal{A}^* -Hopf algebras.*

We will apply this spectral sequence in the case when $X = \Sigma\mathbb{C}P^\infty$. The cohomology ring $H^*(\Sigma\mathbb{C}P^\infty; \mathbb{k})$ has trivial products. This is always true for a suspension, but can hold more generally. We also have for $n \geq 1$,

$$H^{n+1}(\Sigma\mathbb{C}P^\infty; \mathbb{k}) \cong H^n(\mathbb{C}P^\infty; \mathbb{k}),$$

and

$$H^*(\mathbb{C}P^\infty; \mathbb{k}) = \mathbb{k}[x]$$

with $x \in H^2(\mathbb{C}P^\infty; \mathbb{k})$.

Theorem

There is an isomorphism of Hopf algebras

$$\mathrm{Tor}_{H^*(\Sigma\mathbb{C}P^\infty)}^{*,*} = B^*(H^*(\Sigma\mathbb{C}P^\infty)),$$

where $B^*(H^*(\Sigma\mathbb{C}P^\infty))$ denotes the bar construction with

$$B^{-s}(H^*(\Sigma\mathbb{C}P^\infty)) = (\tilde{H}^*(\Sigma\mathbb{C}P^\infty))^{\otimes s}$$

for $s \geq 0$. The coproduct

$$\psi: B^{-s}(H^*(\Sigma\mathbb{C}P^\infty)) \longrightarrow \bigoplus_{i=0}^s B^{-i}(H^*(\Sigma\mathbb{C}P^\infty)) \otimes B^{i-s}(H^*(\Sigma\mathbb{C}P^\infty))$$

is the standard one for which

$$\psi([u_1 | \cdots | u_s]) = \sum_{i=0}^s [u_1 | \cdots | u_i] \otimes [u_{i+1} | \cdots | u_s].$$

Corollary

The Eilenberg-Moore spectral sequence of the Theorem collapses at the E_2 -term.

Here are two more observations on this spectral sequence.

Lemma

The edge homomorphism $e: E_2^{-1,+1} \longrightarrow H^*(\Omega\Sigma\mathbb{C}P^\infty)$ can be identified with the composition*

$$H^{*+1}(\Sigma\mathbb{C}P^\infty) \xrightarrow{\text{ev}^*} H^{*+1}(\Sigma\Omega\Sigma\mathbb{C}P^\infty) \xrightarrow{\cong} H^*(\Omega\Sigma\mathbb{C}P^\infty)$$

using the canonical isomorphism $E_2^{-1,+1} \xrightarrow{\cong} H^{*+1}(\Sigma\mathbb{C}P^\infty)$.*

Corollary

The edge homomorphism $e: E_2^{-1,+1} \longrightarrow H^*(\Omega\Sigma\mathbb{C}P^\infty)$ is a monomorphism.*

To obtain information about products in $H^*(\Omega\Sigma\mathbb{C}P^\infty)$ we will make use of Steenrod operations.

We sketch the argument for $\mathbb{k} = \mathbb{F}_2$ and set $H^*(-) = H^*(-; \mathbb{F}_2)$. We will use the isomorphism $H^{k+1}(\Sigma\mathbb{C}P^\infty) \cong H^k(\mathbb{C}P^\infty)$ to identify an element $\Sigma y \in H^{k+1}(\Sigma\mathbb{C}P^\infty)$ with $y \in H^k(\mathbb{C}P^\infty)$.

Theorem

$H^*(\Omega\Sigma\mathbb{C}P^\infty)$ is a polynomial algebra.

Proof

Consider $[\Sigma x^{k+1}] \in E_2^{-1,2k+3}$. The Steenrod operation Sq^{2k+2} satisfies

$$\begin{aligned} Sq^{2k+2}[\Sigma x^{k+1}] &= [Sq^{2k+2}(\Sigma x^{k+1})] \\ &= [\Sigma x^{2k+2}] \neq 0. \end{aligned}$$

So the element of $H^*(\Omega\Sigma\mathbb{C}P^\infty)$ represented in the spectral sequence by $[\Sigma x^{k+1}]$ has non-trivial square represented by $Sq^{2k+2}[\Sigma x^{k+1}] = [\Sigma x^{2k+2}] \neq 0$. More generally, using the description of the E_2 -term in the Theorem above, we can similarly see that all elements represented in the E_2 -term are non-nilpotent. Thus the algebra generators of $H^*(\Omega\Sigma\mathbb{C}P^\infty)$ are not nilpotent, so by Borel's theorem we can deduce that $H^*(\Omega\Sigma\mathbb{C}P^\infty)$ is a polynomial algebra. □