Polynomial Hopf algebras in Algebra & Topology

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6th May 2014

last updated 07/05/2014

Graded modules

Given a commutative ring $\Bbbk$, a *graded* $\Bbbk$-*module* $M = M_*$ or $M = M^*$ means sequence of $\Bbbk$-modules M_n or M^n. In practise we will always have $M_n = 0$ or $M^n = 0$ whenever $n < 0$ so M is *connective*. We will usually drop the word graded!

If $x \in M_n$ or $x \in M^n$ then n is the *degree* of x and we set $|x| = n$.

It is useful to view an ungraded $\Bbbk$-module N as graded with $N_0 = N = N^0$ and $N_n = 0 = N^0$ whenever $n \neq 0$.

We can form tensor products of such graded modules by setting

$$(M \otimes_{\Bbbk} N)_n = \bigoplus_i M_i \otimes_{\Bbbk} N_{n-i}$$

and so on. We usually write $\otimes$ for $\otimes_{\Bbbk}$.

There is a switch isomorphism $\mathrm{T}\colon M \otimes N \xrightarrow{\cong} N \otimes M$ for which

$$\mathrm{T}(x \otimes y) = (-1)^{|x|\,|y|} y \otimes x.$$

A (*connected*) $\mathbb{k}$-*algebra* A_* or A^* is a connective $\mathbb{k}$-module with $A_0 = \mathbb{k}$ or $A^0 = \mathbb{k}$, and a $\mathbb{k}$-linear *product* $\varphi\colon A \otimes A \longrightarrow A$, i.e., a sequence $\mathbb{k}$-homomorphisms $A_i \otimes A_j \longrightarrow A_{i+j}$, fitting into some commutative diagrams.

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "\varphi\otimes \mathrm{Id}"'] \arrow[dr, "\mathrm{Id}\otimes\varphi"] & \\
A \otimes A \arrow[dr, "\varphi"'] & & A \otimes A \arrow[dl, "\varphi"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{k} \otimes A \arrow[d, "\eta\otimes \mathrm{Id}"'] & A \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[dd, "\mathrm{Id}"'] & A \otimes \mathbb{k} \arrow[d, "\mathrm{Id}\otimes\eta"] \\
A \otimes A \arrow[dr, "\varphi"'] & & A \otimes A \arrow[dl, "\varphi"] \\
 & A &
\end{tikzcd}$$

Here the unit homomorphism $\eta\colon \mathbb{k} \longrightarrow A$ is the inclusion of $\mathbb{k}$ as A_0 or A^0. A is *commutative* if the following diagram commutes.

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\mathrm{T}", "\cong"'] \arrow[dr, "\varphi"'] & & A \otimes A \arrow[dl, "\varphi"] \\
 & A &
\end{tikzcd}$$

Coalgebras

The *dual* notion is that of a (*connected*) *coalgebra*, which is a triple (C, ψ, ε), with C a connected $\Bbbk$-module, $\psi\colon C \longrightarrow C \otimes C$, and $\varepsilon\colon C \longrightarrow \Bbbk$ trivial except in degree $n = 0$ in which case it is an isomorphism, and this data fits into some commutative diagrams.

$$
\begin{tikzcd}
 & C \otimes C \otimes C & \\
C \otimes C \arrow[ur, "\psi\otimes \mathrm{Id}"] & & C \otimes C \arrow[ul, "\mathrm{Id}\otimes\psi"'] \\
 & C \arrow[ul, "\psi"] \arrow[ur, "\varphi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes C \arrow[r, "\cong"] & C & C \otimes \Bbbk \arrow[l, "\cong"'] \\
C \otimes C \arrow[u, "\varepsilon\otimes \mathrm{Id}"] & & C \otimes C \arrow[u, "\mathrm{Id}\otimes\varepsilon"'] \\
 & C \arrow[ul, "\psi"] \arrow[uu, "\mathrm{Id}"] \arrow[ur, "\psi"'] &
\end{tikzcd}
$$

If the following diagram commutes then C is *cocommutative*.

$$
\begin{tikzcd}
C \otimes C \arrow[rr, "\mathrm{T}", "\cong"'] & & C \otimes C \\
 & C \arrow[ul, "\psi"] \arrow[ur, "\psi"'] &
\end{tikzcd}
$$

Examples

Let x have degree $d \in \mathbb{N}$. Then the *free* $\Bbbk$*-algebra* $\Bbbk\langle x\rangle$ has

$$\Bbbk\langle x\rangle_{kd} = \Bbbk\langle x\rangle^{kd} = \Bbbk\{x^k\},$$

and is trivial in degrees not divisible by d. The *free commutative* $\Bbbk$*-algebra* $\Bbbk[x]$ is the quotient algebra $\Bbbk\langle x\rangle/(x^2 - (-1)^d x^d)$. When $\operatorname{char}\Bbbk = 2$, $\Bbbk[x] = \Bbbk\langle x\rangle$, but if $2 \in \Bbbk^\times$ and d is odd, $\Bbbk[x] = \Bbbk\langle x\rangle/(x^2)$ is an *exterior algebra*. This generalises to free commutative algebras on collections of elements x_α of positive degrees. If all generators are in even degrees then we get a *polynomial algebra*

$$\Bbbk[x_\alpha : \alpha] = \bigotimes_\alpha \Bbbk[x_\alpha],$$

if they are all in odd degrees then we get an *exterior algebra*

$$\Bbbk[x_\alpha : \alpha] = \Lambda_\Bbbk(x_\alpha : \alpha).$$

The *free algebra* on a collection of elements y_β is built out of the tensor powers of the free module $Y = \Bbbk\{y_\beta : \beta\}$.

For some basic coalgebras, we can take an indeterminate y of even degree $2d$ and $C = \Bbbk[y]$. For $\psi\colon C \longrightarrow C \otimes C$ take the *Binomial coproduct*

$$\psi(y^k) = \sum_{i=0}^{k} \binom{k}{i} y^i \otimes y^{k-i},$$

and also set $\varepsilon(y^k) = 0^k$.
For a more interesting version, take $C_{2k} = \Bbbk\{y^{[k]}\}$ and the *Leibnitz coproduct*

$$\psi(y^{[k]}) = \sum_{i=0}^{k} y^{[i]} \otimes y^{[k-i]}.$$

If $\operatorname{char} \Bbbk = 0$ then we can think of $y^{[k]}$ as $y^k/k!$, but the above makes sense for any $\Bbbk$.

Hopf algebras

Suppose that (A, φ, η) is an algebra and (A, ψ, ε) is a coalgebra. Then $(A, \varphi, \eta, \psi, \varepsilon)$ is a Hopf algebra if either of the following holds:

- (A, φ, η) is commutative and ψ, ε are algebra homomorphisms;
- (A, ψ, ε) is cocommutative and φ, η are coalgebra homomorphisms.

Note that in the first case φ, η are algebra homomorphisms, while in the second, ψ, ε are coalgebra homomorphisms. Here the tensor product of algebras A_1, A_2 is given the product

$$(A_1 \otimes A_2) \otimes (A_1 \otimes A_2) \xrightarrow{\cong} (A_1 \otimes A_1) \otimes (A_2 \otimes A_2) \xrightarrow{\varphi_1 \otimes \varphi_2} A_1 \otimes A_2$$

and similarly for coalgebras. So these notions are even more symmetric than might appear. The Hopf algebra is called *bicommutative* if both algebra and coalgebra structures are commutative.

If A is a commutative algebra or a commutative coalgebra, the connectivity assumption forces the existence of an *antipode* $\chi\colon A \longrightarrow A$ which is an involution that is both an algebra and a coalgebra anti-isomorphism making the following diagram commutative.

$$
\begin{tikzcd}
 & A \arrow[dl, "\psi"'] \arrow[dd, "\varepsilon"] \arrow[dr, "\psi"] & \\
A \otimes A \arrow[dd, "\chi \otimes \mathrm{Id}"'] & & A \otimes A \arrow[dd, "\mathrm{Id} \otimes \chi"] \\
 & \Bbbk \arrow[dd, "\eta"] & \\
A \otimes A \arrow[dr, "\varphi"'] & & A \otimes A \arrow[dl, "\varphi"] \\
 & A &
\end{tikzcd}
$$

The symmetric function Hopf algebra

Take generators $c_n \in \mathrm{Symm}^{2n}$ and form the polynomial algebra $\mathrm{Symm} = \Bbbk[c_n : n \geqslant 1]$. Notice that the free module $\Bbbk\{c_n : n \geqslant 0\}$ is also a cocommutative coalgebra with the Leibniz coproduct.

Theorem

Symm *is the free bicommutative Hopf algebra generated by the cocommutative coalgebra* $\Bbbk\{c_n : n \geqslant 0\}$*.*

Define the *dual* of Symm by taking the $\Bbbk$-linear dual

$$\mathrm{Symm}_n = \mathrm{Hom}_{\Bbbk}(\mathrm{Symm}^n, \Bbbk)$$

and taking the adjoints φ_*, ψ_* to be the compositions

$$\begin{aligned}\mathrm{Symm}_* \otimes \mathrm{Symm}_* &\xrightarrow{\cong} \mathrm{Hom}_{\Bbbk}(\mathrm{Symm} \otimes \mathrm{Symm}, \Bbbk) \\ &\xrightarrow{\psi^\dagger} \mathrm{Hom}_{\Bbbk}(\mathrm{Symm}, \Bbbk) = \mathrm{Symm}_*,\end{aligned}$$

$$\begin{aligned}\mathrm{Symm}_* = \mathrm{Hom}_{\Bbbk}(\mathrm{Symm}, \Bbbk) &\xrightarrow{\varphi^\dagger} \mathrm{Hom}_{\Bbbk}(\mathrm{Symm} \otimes \mathrm{Symm}, \Bbbk) \\ &\xrightarrow{\cong} \mathrm{Symm}_* \otimes \mathrm{Symm}_*.\end{aligned}$$

We also define $\eta * = \varepsilon^\dagger$ and $\varepsilon^* = \eta^\dagger$. Then $(\mathrm{Symm}_*, \varphi^*, \eta^*, \psi^*, \varepsilon^*)$ is a bicommutative Hopf algebra.

Theorem

There is an isomorphism of Hopf algebras $\mathrm{Symm}^* \cong \mathrm{Symm}_*$*, hence* Symm^* *is self dual.*

Corollary

Symm_* *is a polynomial algebra.*

Under this isomorphism $\mathrm{Symm}^* \cong \mathrm{Symm}_*$, let $c_n \leftrightarrow b_n$. We can also try to understand elements of Symm_* in terms of duality. If we use the monomial basis $c_1^{r_1} \cdots c_\ell^{r_\ell}$ then the dual of the monomial c_1^k is b_n, while the dual of the indecomposable c_n is an element q_n which satisfies $\psi_*(q_n) = q_n \otimes 1 + 1 \otimes q_n$ so it is *primitive*. In fact the primitive module in degree $2n$ is generated by q_n,

$$\mathrm{Pr}\,\mathrm{Symm}_{2n} = \Bbbk\{q_n\}$$

and the Newton recurrence formula is satisfied:

$$q_n = b_1 q_{n-1} - b_2 q_{n-1} + \cdots + (-1)^{n-2} b_{n-1} q_1 + (-1)^{n-1} n b_n.$$

Under the isomorphism there is also a primitive s_n in Symm^{2n}. There is a self dual basis consisting of Schur functions $S_\mu(c_1, \ldots)$ which are dual to the $S_\mu(b_1, \ldots)$. The s_n and q_n are special cases of these.

The structure of Symm is sensitive to the ring $\Bbbk$. For example, if $\Bbbk = \mathbb{Q}$, there is a decomposition of Hopf algebras

$$\mathrm{Symm}^* = \bigotimes_{n \geqslant 1} \mathbb{Q}[s_n].$$

Let p be a prime and let $\Bbbk = \mathbb{F}_p$ or $\Bbbk = \mathbb{Z}_{(p)}$. There is a decomposition of Hopf algebras

$$\mathrm{Symm}^* = \bigotimes_{p \nmid m} \mathrm{B}[2m],$$

where

$$\mathrm{B}[2m] = \Bbbk[a_{m,r} : r \geqslant 0]$$

is an indecomposable polynomial Hopf algebra and

$$s_{mp^r} = p^r a_{m,r} + p^{r-1} a_{m,r-1}^{p} + \cdots + p a_{m,1}^{p^{r-1}} + a_{m,0}^{p^r}.$$

This connection with Witt vectors leads to Symm being viewed as the *big Witt vector* Hopf algebra.

Occurrences of Symm in nature

One interpretation of Symm^{2n} is as the $\Bbbk$-module of homogeneous symmetric functions of degree n in k indeterminates t_i where $k \geqslant n$. It is a classical result that this is correct and then c_n corresponds to the elementary function $\sum t_1 \cdots t_n$, while s_n corresponds to the power sum $\sum t_1^n$.

We can also identify Symm^{2n} with the representation/character ring of the symmetric group Σ_n, $\mathrm{R}(\Sigma_n)$ under addition. Then $\mathrm{R} = \bigoplus_{n \geqslant 0} \mathrm{R}(\Sigma_n)$ has a Hopf algebra structure agreeing with that of Symm and it is also self dual under inner product of characters.

In Algebraic Topology, Symm^* appears as $H^*(BU; \Bbbk)$, the cohomology ring of the classifying space BU. The coproduct is induced from the map $BU \times BU \longrightarrow BU$ classifying direct summand of vector bundles. Here c_n is the universal n-th Chern class and the coproduct is equivalent to the Cartan formula. Dually, $\mathrm{Symm}_* = H_*(BU; \Bbbk)$.

A non-commutative analogue

Starting with the Leibnitz cocommutative coalgebra $\Bbbk\{z_n : n \geqslant 0\}$ where $|z_n| = 2n$. We can form the free algebra generated by the z_n with $n \geqslant 1$, $\mathrm{NSymm}_* = \Bbbk\langle z_n : n \geqslant 1\rangle$. It has a basis consisting of ordered monomials $z_{r_1} \cdots z_{r_\ell}$. The Leibnitz coproduct extends, e.g.,

$$\psi_*(z_m z_n) = \sum_{i,j} z_i z_j \otimes z_{m-i} z_{n-j}.$$

The counit is given by $\varepsilon_*(z_k) = 0$ if $k > 0$ and $\varepsilon_*(z_0) = \varepsilon_*(1) = 1$.

Theorem

NSymm_* *is a cocommutative Hopf algebra.*

The ring of quasi-symmetric functions QSymm^* is the dual, $\mathrm{QSymm}^n = \mathrm{Hom}_{\Bbbk}(\mathrm{NSymm}_n, \Bbbk)$.

Theorem

QSymm^* *is a commutative Hopf algebra.*

Ditters Conjecture ca 1972: QSymm^* is a polynomial ring. (First apparently correct proof by Hazewinkel 2000).

The product in QSymm is complicated. If we denote by $[r_1, \ldots, r_\ell]$ the dual to the monomial $z_{r_1} \cdots z_{r_\ell}$ then products involve *overlapping shuffles*. For example,

$$\begin{aligned}[1,2][3] &= [1,2,3] + [1,3,2] + [3,1,2] + [1,2+3] + [1+3,2] \\ &= [1,2,3] + [1,3,2] + [3,1,2] + [1,5] + [4,2].\end{aligned}$$

In NSymm_* there are many primitives in each degree. For example,

$$\begin{aligned}q'_n &= z_1 q'_{n-1} - z_2 q'_{n-1} + \cdots + (-1)^{n-2} z_{n-1} q'_1 + (-1)^{n-1} n z_n, \\ q''_n &= q''_{n-1} z_1 - q''_{n-1} z_2 + \cdots + (-1)^{n-2} q''_1 z_{n-1} + (-1)^{n-1} n z_n,\end{aligned}$$

define two different families of primitives. This makes it difficult to understand the indecomposables in QSymm^*.

Topology to the rescue!

In fact, these Hopf algebras appear in topological disguise:

$$\mathrm{NSymm}_* \cong H_*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty), \quad \mathrm{QSymm}^* \cong H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty).$$

Theorem (Topological proof: Baker & Richter 2006)
$H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$ *is polynomial.*

I will outline an approach to proving this which differs from the original one but extends to many other examples in a uniform way. To simplify things I'll only concentrate on the case of a field $\Bbbk$, the most interesting example being $\Bbbk = \mathbb{F}_p$ for a prime p. The rational case is easy and we have a local to global argument for the case $\Bbbk = \mathbb{Z}$.

The Eilenberg-Moore spectral sequence

Let $\Bbbk$ be a field and write $H^*(-) = H^*(-;\Bbbk)$. We also need to assume that X is simply connected.

Theorem

There is a second quadrant Eilenberg-Moore spectral sequence of $\Bbbk$-Hopf algebras $(\mathrm{E}_r^{*,*}, d_r)$ with differentials

$$d_r \colon \mathrm{E}_r^{s,t} \longrightarrow \mathrm{E}_r^{s+r,t-r+1}$$

and

$$\mathrm{E}_2^{s,t} = \mathrm{Tor}_{H^*(X)}^{s,t}(\Bbbk,\Bbbk) \Longrightarrow H^{s+t}(\Omega X).$$

The grading conventions here give

$$\mathrm{Tor}_{H^*(X)}^{s,*} = \mathrm{Tor}_{-s,*}^{H^*(X)}$$

in the standard homological grading.

When $\Bbbk = \mathbb{F}_p$ for a prime p, this spectral sequence admits Steenrod operations. We will denote the mod p Steenrod algebra by $\mathcal{A}^* = \mathcal{A}(p)^*$.

Theorem

If $H^*(-) = H^*(-; \mathbb{F}_p)$ for a prime p, then the Eilenberg-Moore spectral sequence is a spectral sequence of $\mathcal{A}^*$-Hopf algebras.

We will apply this spectral sequence in the case when $X = \Sigma\mathbb{C}\mathrm{P}^\infty$. The cohomology ring $H^*(\Sigma\mathbb{C}\mathrm{P}^\infty; \Bbbk)$ has trivial products. This is always true for a suspension, but can hold more generally. We also have for $n \geqslant 1$,

$$H^{n+1}(\Sigma\mathbb{C}\mathrm{P}^\infty; \Bbbk) \cong H^n(\mathbb{C}\mathrm{P}^\infty; \Bbbk),$$

and

$$H^*(\mathbb{C}\mathrm{P}^\infty; \Bbbk) = \Bbbk[x]$$

with $x \in H^2(\mathbb{C}\mathrm{P}^\infty; \Bbbk)$.

Theorem

There is an isomorphism of Hopf algebras

$$\mathrm{Tor}^{*,*}_{H^*(\Sigma\mathbb{C}\mathrm{P}^\infty)} = \mathrm{B}^*(H^*(\Sigma\mathbb{C}\mathrm{P}^\infty)),$$

where $\mathrm{B}^*(H^*(\Sigma\mathbb{C}\mathrm{P}^\infty))$ *denotes the bar construction with*

$$\mathrm{B}^{-s}(H^*(\Sigma\mathbb{C}\mathrm{P}^\infty)) = (\widetilde{H}^*(\Sigma\mathbb{C}\mathrm{P}^\infty))^{\otimes s}$$

for $s \geqslant 0$. *The coproduct*

$$\psi\colon \mathrm{B}^{-s}(H^*(\Sigma\mathbb{C}\mathrm{P}^\infty)) \longrightarrow \bigoplus_{i=0}^{s} \mathrm{B}^{-i}(H^*(\Sigma\mathbb{C}\mathrm{P}^\infty)) \otimes \mathrm{B}^{i-s}(H^*(\Sigma\mathbb{C}\mathrm{P}^\infty))$$

is the standard one for which

$$\psi([u_1|\cdots|u_s]) = \sum_{i=0}^{s} [u_1|\cdots|u_i] \otimes [u_{i+1}|\cdots|u_s].$$

Corollary

The Eilenberg-Moore spectral sequence of the Theorem collapses at the E_2*-term.*

Here are two more observations on this spectral sequence.

Lemma

The edge homomorphism $e\colon \mathrm{E}_2^{-1,*+1} \longrightarrow H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$ *can be identified with the composition*

$$H^{*+1}(\Sigma\mathbb{C}\mathrm{P}^\infty) \xrightarrow{\mathsf{ev}^*} H^{*+1}(\Sigma\Omega\Sigma\mathbb{C}\mathrm{P}^\infty) \xrightarrow{\cong} H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$$

using the canonical isomorphism $\mathrm{E}_2^{-1,*+1} \xrightarrow{\cong} H^{*+1}(\Sigma\mathbb{C}\mathrm{P}^\infty)$.

Corollary

The edge homomorphism $e\colon \mathrm{E}_2^{-1,*+1} \longrightarrow H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$ *is a monomorphism.*

To obtain information about products in $H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$ we will make use of Steenrod operations.

We sketch the argument for $\Bbbk = \mathbb{F}_2$ and set $H^*(-) = H^*(-;\mathbb{F}_2)$. We will use the isomorphism $H^{k+1}(\Sigma\mathbb{C}\mathrm{P}^\infty) \cong H^k(\mathbb{C}\mathrm{P}^\infty)$ to identify an element $\Sigma y \in H^{k+1}(\Sigma\mathbb{C}\mathrm{P}^\infty)$ with $y \in H^k(\mathbb{C}\mathrm{P}^\infty)$.

Theorem

$H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$ *is a polynomial algebra.*

Proof

Consider $[\Sigma x^{k+1}] \in \mathrm{E}_2^{-1,2k+3}$. The Steenrod operation Sq^{2k+2} satisfies

$$\begin{aligned}\mathrm{Sq}^{2k+2}[\Sigma x^{k+1}] &= [\mathrm{Sq}^{2k+2}(\Sigma x^{k+1})] \\ &= [\Sigma x^{2k+2}] \neq 0.\end{aligned}$$

So the element of $H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$ represented in the spectral sequence by $[\Sigma x^{k+1}]$ has non-trivial square represented by $\mathrm{Sq}^{2k+2}[\Sigma x^{k+1}] = [\Sigma x^{2k+2}] \neq 0$. More generally, using the description of the E_2-term in the Theorem above, we can similarly see that all elements represented in the E_2-term are non-nilpotent. Thus the algebra generators of $H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$ are not nilpotent, so by Borel's theorem we can deduce that $H^*(\Omega\Sigma\mathbb{C}\mathrm{P}^\infty)$ is a polynomial algebra. $\square$